

USE OF A GENERALIZED SIGMOID GROWTH FUNCTION TO PREDICT SITE INDEX FOR UNMANAGED LOBLOLLY AND SLASH PINE PLANTATIONS IN EAST TEXAS

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Abstract—A generalized sigmoid growth function was used in this study to model site index (SI) for unmanaged or low-intensity managed loblolly pine (*Pinus taeda*, L.) and slash pine (*Pinus elliotii*, Engelm.) plantations in east Texas. Schnute's growth function was fit to 11,367 and 5,040 height-age observations of loblolly and slash pine, respectively. These data were collected over a 20-year period from unmanaged pine plantations located across the east Texas region as a part of the East Texas Pine Plantation Research Project (ETPPRP). The fit model was used to derive SI equations (25-year index age). The SI equations apply to unmanaged or low-intensity managed loblolly and slash pine plantations in east Texas ranging in age from 5 to 40 years. They can also serve as a baseline for SI estimation of intensively managed pine plantations.

INTRODUCTION

Forestland in east Texas occupies about 12.1 million acres (Miles 2005). Of this area, 2.9 million acres are classified as pine plantations, with about 2.7 million acres (90 percent) of this total on private land. Estimation of the productivity of these plantations is vitally important to forecasting future wood fiber yields. Site index (SI) is the most common measure used to assess a site's productivity (Clutter and others 1983). Sigmoid growth functions have been used for decades to predict SI (Pienaar and Turnbull 1973). The Chapman-Richards (Chapman 1961, Richards 1959) and Von Bertalanffy (Von Bertalanffy 1951) functions represent two sigmoid functions commonly used as guide curves to develop anamorphic families of site curves (Clutter and others 1983). Schnute (1981) generalized these sigmoid growth functions, with specific application to fish growth. This function is highly flexible in that it can represent many aspects of biological growth, such as asymptotic growth or the lack thereof. In fact, many widely used growth functions, such as the Chapman-Richards and Logistic growth functions, are special cases of Schnute's growth function. The Schnute growth function has found application in forestry for modeling individual tree diameter and basal area growth (Bredenkamp and Gregoire 1988, Yuancai and others 1997), stand density and yield (Zhang and others 1993), and individual tree height-diameter (Peng and others 2001, Zhang 1997). However, to the best of our knowledge, the Schnute growth function has not been applied in the development of SI curves.

The purpose of this study was to use the Schnute growth function to develop a family of anamorphic SI curves for loblolly pine and slash pine in east Texas. These new curves were then compared to existing site curves for loblolly and slash pine in east Texas.

METHODS

Background Information – Chapman-Richards Growth Function

The Chapman-Richards growth function is based on the first-order ordinary differential equation:

$$\frac{dY}{dt} = \alpha Y^{\beta} - \gamma Y \quad (1)$$

or

$$\frac{dY}{dt} + \gamma Y = \alpha Y^{\beta}$$

where

Y = size of organism

t = time

$\frac{dY}{dt}$ = growth of organism

αY^{β} = anabolic growth (e.g., photosynthesis)

γY = catabolic growth (e.g., respiration)

Equation (1) is a Bernoulli equation that can be solved with traditional methods (Grossman and Derrick 1988):

$$Y(t) = \phi \left(1 - e^{-\theta(t-t_0)} \right)^{1/(1-\beta)} \quad (2)$$

where

Y(t) = size of organism at time t

t_0 = time zero or beginning time

$$\phi = \left(\frac{\alpha}{\gamma} \right)^{1/(1-\beta)}$$

$$\theta = \gamma(1 - \beta)$$

all other variables defined above.

A more familiar formulation of equation (2) is the empirical 3-parameter Chapman-Richards growth function:

$$Y(t) = b_0 \left(1 - e^{-b_1(t-t_0)} \right)^{b_2}, \quad (3)$$

where

b_i = regression parameters to be estimated

all other variables are defined as before.

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Background Information –

Schnute Growth Function

The Schnute growth function is based on two first-order ordinary differential equations:

$$\frac{dY}{dt} = YZ \text{ and } \frac{dZ}{dt} = -Z(a + (1-b)Z),$$

where

Z = growth rate

a, b = constants

all other variables defined as before.

Thus, $\frac{dY}{dt}$ states that the change in size (i.e., growth) is a function of size, Y, and the growth rate, Z; and, $\frac{dZ}{dt}$ states

that the change in the growth rate, Z, is a linear function of Z. Together, they give the second-order ordinary differential equation that describes the acceleration of growth:

$$\frac{d^2Y}{dt^2} = \frac{dY}{dt}(-a + (1-b)Z)$$

where

all variables are defined above.

Solution of this second-order differential equation gives:

$$Y(t) = \left(y_1^b + (y_2^b - y_1^b) \frac{1 - e^{-a(t-\tau_1)}}{1 - e^{-a(\tau_2-\tau_1)}} \right)^{1/b} \quad (4)$$

where

Y(t) = size of organism at time t

y_1, y_2 = size of organism at τ_1 and τ_2 (to be estimated via regression)

τ_1, τ_2 = ages at time 1 and 2 (e.g., old and young)

a, b = constants to be estimated via regression $\neq 0$.

Equation (4) is based on acceleration of growth, not just growth as in other models such as Chapman-Richards. Depending on the values of a and b, equation (4) takes on different forms, where some are asymptotic and others are non-asymptotic. In any event, each case is a limiting form of one function. The Chapman-Richards, Von Bertalanffy, Gompertz, Exponential, and Logistic growth functions represent special cases of certain limiting forms, all found by algebraic tinkering of the Schnute function. This study is concerned with Schnute's Case 1 [$a \neq 0, b \neq 0$, equation (4)], because site curves are typically asymptotic.

Guide Curve Development – Chapman-Richards Growth Function

The guide curve method is used to develop anamorphic SI curves (Clutter and others 1983). This method first specifies SI in terms of a mathematical function. In this study, SI is defined as the average height of the 10 tallest trees at the index age of 25 years. For the Chapman-Richards growth function (equation 3), this can be expressed as:

$$S = b_0 \left(1 - e^{-b_1(t_{IA} - t_0)} \right)^{b_2}$$

where

S = SI in feet

t_{IA} = index age = 25 years

t_0 = time zero = 0

all other variables defined as before.

Then, solve for b_0 :

$$b_0 = S \left(1 - e^{-b_1(t_{IA} - t_0)} \right)^{-b_2}.$$

Substituting this into equation (3) gives:

$$H = S \left(\frac{1 - e^{-b_1(t)}}{1 - e^{-b_1(t_{IA})}} \right)^{b_2} \quad (5)$$

and then solving for S gives

$$S = H \left(\frac{1 - e^{-b_1(t_{IA})}}{1 - e^{-b_1(t)}} \right)^{b_2} \quad (6)$$

where

H = Y(t) = average height of the tallest 10 trees in feet at time t

all other variables are defined as before.

Equations (5) and (6) represent a family of anamorphic SI curves described by the Chapman-Richards growth function.

Guide Curve Development – Schnute Growth Function

The methodology outlined in the preceding section was used to develop anamorphic SI curves described by the Schnute growth function, equation (4). First, define the guide curve:

$$S = \left(y_1^b + (y_2^b - y_1^b) \frac{1 - e^{-a(t_{IA} - \tau_1)}}{1 - e^{-a(\tau_2 - \tau_1)}} \right)^{1/b}.$$

Then, solve for y_2^b :

$$y_2^b = y_1^b + \left(S^b - y_1^b \right) \frac{1 - e^{-a(\tau_2 - \tau_1)}}{1 - e^{-a(t_{IA} - \tau_1)}}$$

Substituting this expression in equation (4) gives:

$$H = \left(y_1^b + \left(S^b - y_1^b \right) \frac{1 - e^{-a(t - \tau_1)}}{1 - e^{-a(t_{IA} - \tau_1)}} \right)^{1/b} \quad (7)$$

and then solving for S gives,

$$S = \left(y_1^b + \left(H^b - y_1^b \right) \frac{1 - e^{-a(t_{IA} - \tau_1)}}{1 - e^{-a(t - \tau_1)}} \right)^{1/b} \quad (8)$$

where

all variables are defined as before.

Equations (7) and (8) represent a family of anamorphic SI curves described by the Schnute growth function.

Data Analysis

Currently, 124 permanent plots are located in loblolly pine plantations, and 56 plots are located in slash pine plantations throughout east Texas. The ETPPRP study area covers 22 counties across east Texas (Lenhart and others 1985). Generally, the counties are located within the rectangle from 30 to 35 north latitude and 93 to 96 west longitude. Each plot consists of two subplots: one for model development and one for model evaluation. A subplot is 100 x 100 feet in size, and a 60-foot buffer separates the subplots. All planted pine trees are permanently tagged and numbered. Only the model development plots were used in this study. The average height of the 10 tallest site trees and the total age of the plantation were used to represent height and age in the functions. The 10 tallest trees per plot were considered site trees if they met the following criteria: (1) free of damage, (2) no forks, and (3) no presence of stem fusiform rust [*Cronartium quercuum* [Berk.] Miyabe ex Shirai f. sp. *Fusiforme*]. A total of 11,367 height-age observations for loblolly pine and 5,040 height-age observations for slash pine (table 1) were used to fit equation (4). PROC NLIN in SAS version 9.1 was used to run the analyses.

RESULTS

Equation (4) was fit to the loblolly and slash pine data to produce the coefficients for equations (7) and (8). All coefficients were significantly different from zero (table 2), and the residual plots did not reveal any unusual heteroscedasticity problems (plots not shown). Note that τ_1 and τ_2 were fixed at the values of the youngest and oldest ages, while a , b , y_1 , and y_2 were estimated by SAS. The SAS code fragment used to fit the Schnute model is provided in the appendix. The coefficient values from table 2 were used in equation (8) to produce site curves for loblolly pine (fig. 1) and slash pine (fig. 2). These curves range in SI from 40 to 90 feet (index age = 25 years), and they apply to plantations that range from 5 to 40 years of age.

The Schnute guide curves differ significantly from those of Lenhart and others (1986). For loblolly pine, the Schnute curve was higher than Lenhart and others' curve for all ages (fig. 3). For slash pine, the Schnute curve was higher for ages > 15 years (fig. 4). Since the Chapman-Richards growth function (equation 3) is a special case of the Schnute growth function (equation 4), the Schnute guide curve equations can be converted to the same functional form as those of Lenhart and others, so that the coefficients can be compared using a one-sample t-test. To find asymptotic height, b_0 , of the Chapman-Richards function for loblolly pine, insert parameter values into the Schnute function and let $t \rightarrow \infty$:

Table 1—Descriptive statistics for the ETPPRP loblolly and slash pine development plots, where age = total age (years) of plantation and height = average height (feet) of the 10 tallest site trees on a plot

Species	Variable	N	Mean	Standard deviation	Minimum	Maximum
Loblolly	Age	11,367	14	7	1	37
	Height	11,367	44	21	1	101
Slash	Age	5,040	14	7	1	33
	Height	5,040	43	21	2	97

Table 2—Parameter estimates and fit statistics of loblolly and slash pine guide curves (equation 4)

Species	Parameter	Parameter estimate	Standard error	Lower 95% confidence interval	Upper 95% confidence interval	Root MSE
feet						
Loblolly	y_1	0.99476	0.36757	0.27432	1.71520	7.3
	y_2	80.92198	0.48019	79.98081	81.86315	
	a	0.08036	0.00342	0.07366	0.08706	
	b	0.68232	0.04006	0.60380	0.76084	
	τ_1	1	—	—	—	
	τ_2	37	—	—	—	
Slash	y_1	3.14282	0.44793	2.27034	4.02622	6.7
	y_2	82.25850	0.66399	80.95708	83.55992	
	a	0.07747	0.00554	0.06661	0.08833	
	b	0.52098	0.06477	0.39403	0.64793	
	τ_1	1	—	—	—	
	τ_2	33	—	—	—	

— = na.

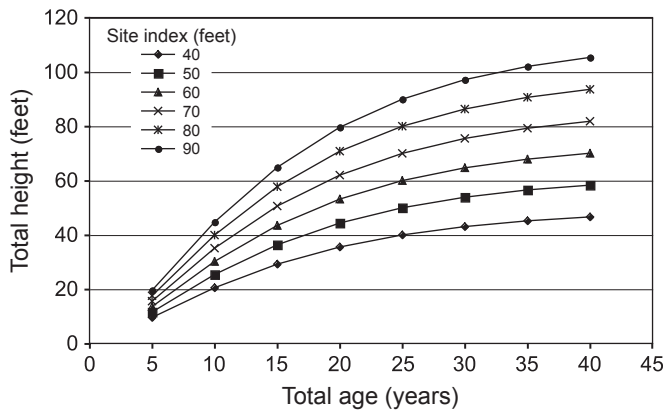


Figure 1—Site index curves (index age = 25 years) for unmanaged loblolly pine plantations in east Texas.

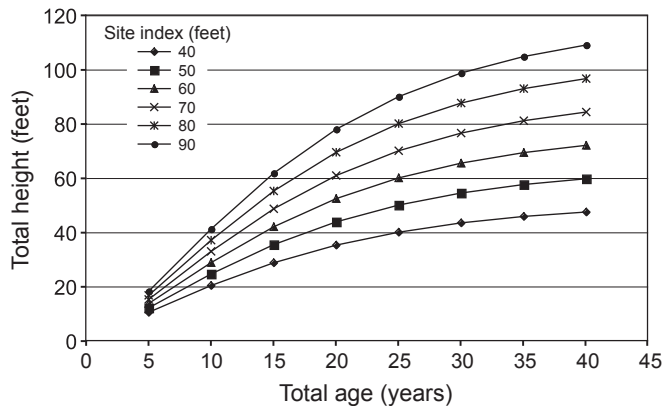


Figure 2—Site index curves (index age = 25 years) for unmanaged slash pine plantations in east Texas.

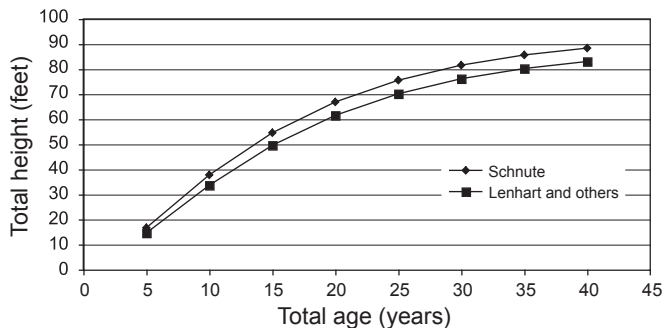


Figure 3—Comparison between the new Schnute guide curve and the Lenhart and others (1985) guide curve for loblolly pine.

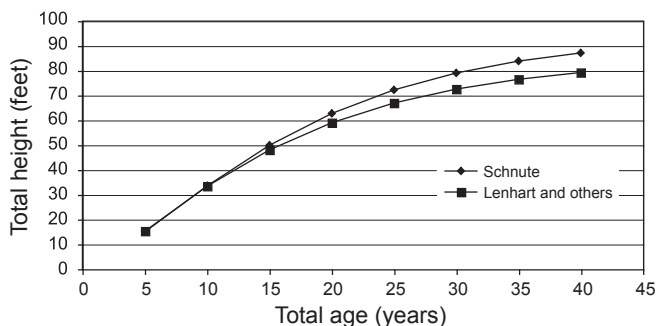


Figure 4—Comparison between the new Schnute guide curve and the Lenhart and others (1985) guide curve for slash pine.

$$b_0 = \left(0.99476^{0.68232} + (80.92198^{0.68232} - 0.99476^{0.68232}) \left(\frac{1 - 0}{1 - e^{-0.08036(37-1)}} \right)^{1/0.68232} \right)^{1/0.68232} = 87.6214 \quad (8)$$

Similarly, find the values for the other two coefficients:

$$b_1 = a = 0.08036, \text{ and}$$

$$b_2 = 1 / b = 1 / 0.68232 = 1.4656.$$

Thus, the Chapman-Richards guide curve function derived from equation (4) for loblolly pine is:

$$\text{Height} = 87.6214 * \left(1 - e^{-0.08036 * (\text{Age} - 0)} \right)^{1.4656}.$$

The same procedure applied to equation (4) for slash pine gives:

$$\text{Height} = 94.4730 * \left(1 - e^{-0.07747 * (\text{Age} - 0)} \right)^{1.9195}.$$

The t-test revealed that the b_2 coefficient for the loblolly pine equation is significantly different ($p < 0.0001$) than that of Lenhart and others (1986). Furthermore, the b_0 and b_2 coefficients for the slash pine equation are significantly different than those of Lenhart and others (1986) $p = 0.0149$ and $p < 0.0001$, respectively. These results further support the claim that Lenhart and others' guide curves are significantly lower than the Schnute guide curves. This leads to an underestimation of average height for the trees in this dataset, which will underestimate SI. Thus, the new Schnute SI equations/curves represent an improvement over those of Lenhart and others (1986). The additional height-age data now available from older stands that encompass the index age of 25 years are most likely responsible for the improvement in SI estimation by the new Schnute SI curves.

CONCLUSIONS AND RECOMMENDATIONS

The Schnute growth function can be applied to many forestry prediction problems: diameter and basal area growth, height-age, density, yield, and SI. The Schnute growth function can also be used as a guide curve to develop a family of anamorphic SI curves, either in its original form (equation 4) or in the Chapman-Richards form (equation 3). This study provides new SI curves and equations for unmanaged or low-intensity managed loblolly and slash pine plantations in east Texas. These new curves are an improvement over Lenhart and others (1986) because in this study, height-age data for model fitting encompass the 25-year index age.

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APPENDIX

The following SAS code fragment was used to estimate coefficients in equation (4) for loblolly pine:

```
PROC NLIN DATA = LOB_HTAGE_PAIRS OUTEST = TEST;
```

```
MODEL HEIGHT = (B0**B2 + (B1**B2 - B0**B2) * ( (1-EXP(-B3 * (AGE-1))) / (1-EXP(-B3 * (37-1))) )) ** (1 / B2);
```

```
PARMS B0 = 5.0
```

```
      B1 = 50.0
```

```
      B2 = 0.8
```

```
      B3 = 0.05;
```